

# Weighted inequalities for quasilinear integral operators on the semiaxis and application to the Lorentz spaces

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**Abstract:** Weighted  $L^p - L^r$  inequalities with arbitrary measurable non-negative weights for positive quasilinear integral operators with Oinarov's kernel on the semiaxis are characterized. Application to the boundedness of maximal operator in the Lorentz  $\Gamma$ -spaces is given.

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## 1 Introduction

Let  $\mathbb{R}_+ := [0, \infty)$ . Denote  $\mathfrak{M}$  the set of all measurable functions on  $\mathbb{R}_+$  and  $\mathfrak{M}^+ \subset \mathfrak{M}$  the subset of all non-negative functions. If  $0 < p \leq \infty$  and  $v \in \mathfrak{M}^+$  we define

$$L_v^p := \left\{ f \in \mathfrak{M} : \|f\|_{L_v^p} := \left( \int_0^\infty |f(x)|^p v(x) dx \right)^{\frac{1}{p}} < \infty \right\},$$

$$L_v^\infty := \left\{ f \in \mathfrak{M} : \|f\|_{L_v^\infty} := \operatorname{ess\,sup}_{x \geq 0} v(x) |f(x)| < \infty \right\}.$$

Let  $0 < q \leq \infty$  and  $w \in \mathfrak{M}^+$ . We consider quasilinear operators on  $\mathfrak{M}^+$  of the form

$$(Tf)(x) = \left( \int_x^\infty w(y) \left( \int_0^y k(y, z) f(z) dz \right)^q dy \right)^{\frac{1}{q}},$$

$$(\mathcal{T}f)(x) = \left( \int_0^x w(y) \left( \int_y^\infty k(z, y) f(z) dz \right)^q dy \right)^{\frac{1}{q}},$$

$$(Sf)(x) = \left( \int_x^\infty w(y) \left( \int_y^\infty k(z, y) f(z) dz \right)^q dy \right)^{\frac{1}{q}},$$

$$(\mathcal{S}f)(x) = \left( \int_0^x w(y) \left( \int_0^y k(y, z) f(z) dz \right)^q dy \right)^{\frac{1}{q}}$$

and

$$(\mathbf{T}f)(x) = \left( \int_x^\infty k(y, x) w(y) \left( \int_0^y f(z) dz \right)^q dy \right)^{\frac{1}{q}},$$

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$$\begin{aligned}
(\mathfrak{T}f)(x) &= \left( \int_0^x k(x,y)w(y) \left( \int_y^\infty f(z)dz \right)^q dy \right)^{\frac{1}{q}}, \\
(\mathbf{S}f)(x) &= \left( \int_x^\infty k(y,x)w(y) \left( \int_y^\infty f(z)dz \right)^q dy \right)^{\frac{1}{q}}, \\
(\mathfrak{S}f)(x) &= \left( \int_0^x k(x,y)w(y) \left( \int_0^y f(z)dz \right)^q dy \right)^{\frac{1}{q}},
\end{aligned}$$

where  $k(x,y) \geq 0$  is a measurable kernel and the right hand sides are to replace by essential supremums

$$\begin{aligned}
(Tf)(x) &= \operatorname{ess\,sup}_{y \geq x} w(y) \int_0^y k(y,z)f(z)dz, \\
(\mathbf{T}f)(x) &= \operatorname{ess\,sup}_{y \geq x} k(y,x)w(y) \int_0^y f(z)dz,
\end{aligned}$$

and similarly for the others, when  $q = \infty$ .

Let  $u, v, w \in \mathfrak{M}^+$  be weights,  $1 \leq p \leq \infty$ ,  $0 < r \leq \infty$ . Our aim is to characterize the weighted inequalities

$$\|Tf\|_{L_u^r} \leq C_T \|f\|_{L_v^p}, \quad f \in \mathfrak{M}^+, \quad (1.1)$$

$$\|\mathcal{T}f\|_{L_u^r} \leq C_{\mathcal{T}} \|f\|_{L_v^p}, \quad f \in \mathfrak{M}^+, \quad (1.2)$$

$$\|Sf\|_{L_u^r} \leq C_S \|f\|_{L_v^p}, \quad f \in \mathfrak{M}^+, \quad (1.3)$$

$$\|\mathcal{S}f\|_{L_u^r} \leq C_{\mathcal{S}} \|f\|_{L_v^p}, \quad f \in \mathfrak{M}^+ \quad (1.4)$$

and

$$\|\mathbf{T}f\|_{L_u^r} \leq C_{\mathbf{T}} \|f\|_{L_v^p}, \quad f \in \mathfrak{M}^+, \quad (1.5)$$

$$\|\mathfrak{T}f\|_{L_u^r} \leq C_{\mathfrak{T}} \|f\|_{L_v^p}, \quad f \in \mathfrak{M}^+, \quad (1.6)$$

$$\|\mathbf{S}f\|_{L_u^r} \leq C_{\mathbf{S}} \|f\|_{L_v^p}, \quad f \in \mathfrak{M}^+, \quad (1.7)$$

$$\|\mathfrak{S}f\|_{L_u^r} \leq C_{\mathfrak{S}} \|f\|_{L_v^p}, \quad f \in \mathfrak{M}^+, \quad (1.8)$$

where a Borel function  $k(x,y) \geq 0$  on  $[0,\infty)^2$  satisfies Oinarov's condition:  $k(x,y) = 0$  if  $x < y$ , and there is a constant  $D \geq 1$  independent of  $x \geq z \geq y \geq 0$  such that

$$\frac{1}{D} (k(x,z) + k(z,y)) \leq k(x,y) \leq D (k(x,z) + k(z,y)) \quad (1.9)$$

and the constants  $C_T$  and others are taken as the least possible. If  $q = r < \infty$  these inequalities are reduced to the generalized Hardy-type inequalities which were well studied see, for instance, [2], [21], [36] with further extensions and improvements in [19], [20], [22], [23], [26], [40], [41] and others. The case  $q = \infty$  is closely related to recently initiated studies of supremum operators [12], [13], [24], [25], [27], [29], [37]. If  $k(x,y) \equiv 1$  the inequality (1.4) plays an important role in analysis on the Morry-type spaces (see, [3], [4], [5], [6], [7]. In particular, for some parameters  $p, q, r$  this case of (1.2) was solved in [16], [17] and (1.4) in [7]. Complete solution of this case is given in [30], [31].

By a new method we characterize the inequalities (1.1)–(1.8) with a kernel  $k(x,y)$  satisfying (1.9) for all parameters  $1 \leq p \leq \infty, 0 < r \leq \infty, 0 < q \leq \infty$ . The cases  $p = \infty$  and  $r = \infty$  are trivial and the interval  $0 < p < 1$  is excluded because in this case it can be shown that if, say,  $C_T < \infty$ , then  $C_T = 0$  (see [28], Theorem 2 for details).

Sections 2 and 3 are devoted to the study of (1.1)–(1.4) and sections 4 and 5 to (1.5)–(1.8). It is interesting to observe that the second part is partially based on the first. In the last section 6 we illustrate our results by a solution of well known problem on a sharp characterization of the  $\Gamma^p(v) \rightarrow \Gamma^q(w)$  boundedness of the Hardy-Littlewood maximal operator for all  $0 < p, q < \infty$  including the most difficult cases missed in [10] and [33].

We use signs  $:=$  and  $=:$  for determining new quantities and  $\mathbb{Z}$  for the set of all integers. For positive functionals  $F$  and  $G$  we write  $F \lesssim G$ , if  $F \leq cG$  with some positive constant  $c$ , which depends only on irrelevant parameters.  $F \approx G$  means  $F \lesssim G \lesssim F$  or  $F = cG$ .  $\chi_E$  denotes the characteristic function (indicator) of a set  $E$ . Uncertainties of the form  $0 \cdot \infty$ ,  $\frac{\infty}{\infty}$  and  $\frac{0}{0}$  are taken to be zero.  $\square$  stands for the end of a proof.

## 2 Operators $T$ and $S$

Suppose for simplicity that  $\int_0^t u < \infty$  for all  $t > 0$  and define the functions  $\sigma : [0, \infty) \rightarrow [0, \infty]$ ,  $\sigma^{-1} : [0, \infty) \rightarrow [0, \infty)$  by (here  $\inf \emptyset = \infty$ )

$$\sigma(x) := \inf \left\{ y > 0 : \int_0^y u \geq 2 \int_0^x u \right\}, \quad \sigma^{-1}(x) := \inf \left\{ y > 0 : \int_0^y u \geq \frac{1}{2} \int_0^x u \right\}.$$

The functions  $\sigma$  and  $\sigma^{-1}$  are increasing and from the continuity of an integral with respect to an upper limit it follows for any  $x \in [0, \infty)$  that  $\int_0^{\sigma^{-1}(x)} u = \frac{1}{2} \int_0^x u$  and if  $\sigma(x) < \infty$ , then  $\int_0^{\sigma(x)} u = 2 \int_0^x u$ .

Let  $\sigma^m$ ,  $m \in \mathbb{N}$  be a composition of  $m$  functions  $\sigma$  and similar for  $\sigma^{-m}$ .

For  $0 < c < d \leq \infty$  and  $f \in \mathfrak{M}^+$  we put

$$(H_{c,d}f)(x) := \chi_{[c,d)}(x) \int_{\sigma^{-1}(c)}^x k(x,z)f(z)dz,$$

$$(H_cf)(x) := \chi_{[c,\infty)}(x) \int_0^x k(x,z)f(z)dz.$$

**Theorem 2.1.** *Let  $1 \leq p < \infty$ ,  $0 < r < \infty$ ,  $0 < q \leq \infty$ ,  $\frac{1}{s} := \left(\frac{1}{r} - \frac{1}{p}\right)_+$ . For validity of the inequality (1.1) it is necessary and sufficient that the inequalities*

$$\left( \int_0^\infty u(x) \left( \int_x^\infty w \right)^{\frac{r}{q}} \left( \int_0^x k(x,z)f(z)dz \right)^r dx \right)^{\frac{1}{r}} \leq A_0 \|f\|_{L_v^p}, \quad (2.1)$$

$$\left( \int_0^\infty u(x) \left( \int_x^\infty [k(z,x)]^q w(z)dz \right)^{\frac{r}{q}} \left( \int_0^x f \right)^r dx \right)^{\frac{1}{r}} \leq A_1 \|f\|_{L_v^p}, \quad (2.2)$$

if  $q < \infty$  or

$$\left( \int_0^\infty u(x) [\text{ess sup}_{y \geq x} w(y)]^r \left( \int_0^x k(x,z)f(z)dz \right)^r dx \right)^{\frac{1}{r}} \leq A_0 \|f\|_{L_v^p}, \quad (2.3)$$

$$\left( \int_0^\infty u(x) [\text{ess sup}_{y \geq x} [w(y)k(y,x)]]^r \left( \int_0^x f \right)^r dx \right)^{\frac{1}{r}} \leq A_1 \|f\|_{L_v^p} \quad (2.4)$$

for  $q = \infty$  hold for all  $f \in \mathfrak{M}^+$  and the constant

$$A_2 := \begin{cases} \sup_{t \in (0, \infty)} \left( \int_0^t u \right)^{\frac{1}{r}} \|H_t\|_{L_v^p \rightarrow L_w^q}, & p \leq r, \\ \left( \int_0^\infty u(x) \left( \int_0^x u \right)^{\frac{s}{p}} \|H_{\sigma^{-1}(x), \sigma(x)}\|_{L_v^p \rightarrow L_w^q}^s dx \right)^{\frac{1}{s}}, & r < p \end{cases} \quad (2.5)$$

is finite. Moreover,  $C_T \approx A_0 + A_1 + A_2$ .

*Proof.* Let  $n_0 \in \mathbb{Z}$  be such an integer that  $2^{n_0} < \int_0^\infty u$ . Put

$$\begin{aligned} a_{n_0} &:= \inf \left\{ y > 0 : \int_0^y u \geq 2^{n_0} \right\}, \\ a_{n+1} &:= \sigma(a_n) \text{ for } n \geq n_0, \\ a_{n-1} &:= \sigma^{-1}(a_n) \text{ for } n \leq n_0. \end{aligned}$$

Denote  $N := \sup\{n \in \mathbb{Z} : a_n < \infty\}$ . If  $N < \infty$  we put  $a_{N+1} := \infty$ . Observe, that  $a_{n-1} = \sigma^{-1}(a_n)$  and  $\sigma(a_n) = a_{n+1}$  for all  $n \leq N$ .

We suppose first that  $q < \infty$ .

*Sufficiency.* Let  $\Delta_n := [a_n, a_{n+1})$ . Applying the condition (1.9) and the relation ([10], Proposition 2.1)

$$\sum_{n \in \mathbb{Z}} 2^n \left( \sum_{i \geq n} \lambda_i \right)^s \approx \sum_{n \in \mathbb{Z}} 2^n \lambda_n^s, \quad (2.6)$$

which is valid for all sequences  $\{\lambda_n\}$  of non-negative numbers and any  $s > 0$ , we have

$$\begin{aligned} \int_0^\infty [Tf]^r u &= \sum_{n \leq N} \int_{\Delta_n} [Tf]^r u \approx \sum_{n \leq N} 2^n \left( \int_{a_n}^\infty w(y) \left( \int_0^y k(y, z) f(z) dz \right)^q dy \right)^{\frac{r}{q}} \\ &\approx \sum_{n \leq N} 2^n \left( \int_{\Delta_n} w(y) \left( \int_0^y k(y, z) f(z) dz \right)^q dy \right)^{\frac{r}{q}} \\ &\approx \sum_{n \leq N} 2^n \left( \int_{\Delta_n} w(y) \left( \int_{a_{n-1}}^y k(y, z) f(z) dz \right)^q dy \right)^{\frac{r}{q}} \\ &+ \sum_{n \leq N} 2^n \left( \int_{\Delta_n} w(y) \left( \int_0^{a_{n-1}} k(y, z) f(z) dz \right)^q dy \right)^{\frac{r}{q}} =: J_1^r + J_2^r. \end{aligned}$$

Since  $k(y, z) \approx k(y, x) + k(x, z)$  for  $y \in \Delta_n$ ,  $x \in \Delta_{n-1}$ ,  $z \in (0, a_{n-1})$ , then  $J_2^r$  is estimated as

follows

$$\begin{aligned}
J_2^r &\approx \sum_{n \leq N} \int_{a_{n-1}}^{a_n} u(x) dx \left( \int_{\Delta_n} w(y) \left( \int_0^{a_{n-1}} k(y, z) f(z) dz \right)^q dy \right)^{\frac{r}{q}} \\
&\approx \sum_{n \leq N} \int_{a_{n-1}}^{a_n} u(x) \left( \int_{a_n}^{a_{n+1}} w(y) [k(y, x)]^q dy \right)^{\frac{r}{q}} dx \left( \int_0^{a_{n-1}} f \right)^r \\
&+ \sum_{n \leq N} \int_{a_{n-1}}^{a_n} u(x) \left( \int_0^{a_{n-1}} k(x, z) f(z) dz \right)^r dx \left( \int_{\Delta_n} w \right)^{\frac{r}{q}} \\
&\lesssim \sum_{n \leq N} \int_{a_{n-1}}^{a_n} u(x) \left( \int_x^\infty w(y) [k(y, x)]^q dy \right)^{\frac{r}{q}} \left( \int_0^x f \right)^r dx \\
&+ \sum_{n \leq N} \int_{a_{n-1}}^{a_n} u(x) \left( \int_x^\infty w \right)^{\frac{r}{q}} \left( \int_0^x k(x, z) f(z) dz \right)^r dx \\
&\lesssim (A_1^r + A_0^r) \|f\|_{L_v^p}^r.
\end{aligned}$$

For an upper bound of  $J_1^r$  we write

$$\begin{aligned}
J_1^r &\approx \sum_{n \leq N} 2^n \|H_{a_n, a_{n+1}} f\|_{L_w^q}^r \\
&\lesssim \sum_{n \leq N} \left( \int_{a_{n-1}}^{a_n} u \right) \|H_{a_n, a_{n+1}}\|_{L_v^p \rightarrow L_w^q}^r \left( \int_{a_{n-1}}^{a_{n+1}} f^p v \right)^{\frac{r}{p}}.
\end{aligned}$$

If  $p \leq r$  we apply Jensen's inequality and get

$$J_1 \lesssim \sup_{n \leq N} \left( \int_{a_{n-1}}^{a_n} u \right)^{\frac{1}{r}} \|H_{a_n, a_{n+1}}\|_{L_v^p \rightarrow L_w^q} \|f\|_{L_v^p} \leq A_2 \|f\|_{L_v^p}.$$

If  $r < p$  we apply Hölder's inequality with the exponents  $\frac{s}{r}$  and  $\frac{p}{r}$  and obtain

$$\begin{aligned}
J_1^s &\lesssim \sum_{n \leq N} \left( \int_{a_n}^{a_{n+1}} u \right)^{\frac{s}{r}} \|H_{a_n, a_{n+1}}\|_{L_v^p \rightarrow L_w^q}^s \|f\|_{L_v^p}^s \\
&\lesssim \sum_{n \leq N} \left( \int_{a_n}^{a_{n+1}} u \right) \left( \int_0^{a_n} u \right)^{\frac{s}{p}} \|H_{\sigma^{-1}(a_{n+1}), \sigma(a_n)}\|_{L_v^p \rightarrow L_w^q}^s \|f\|_{L_v^p}^s \\
&\leq \sum_{n \leq N} \left( \int_{a_n}^{a_{n+1}} u(x) \left( \int_0^x u \right)^{\frac{s}{p}} \|H_{\sigma^{-1}(x), \sigma(x)}\|_{L_v^p \rightarrow L_w^q}^s dx \right) \|f\|_{L_v^p}^s \\
&\leq A_2^s \|f\|_{L_v^p}^s.
\end{aligned}$$

Thus,

$$\|Tf\|_{L_u^r} \lesssim (A_0 + A_1 + A_2) \|f\|_{L_v^p}$$

and the upper bound  $C_T \lesssim A_0 + A_1 + A_2$  is proved.

*Necessity.* Since

$$\begin{aligned} (Tf)(x) &\geq \left( \int_x^\infty w(y) \left( \int_0^x k(y, z) f(z) dz \right)^q dy \right)^{\frac{1}{q}} \\ &\gtrsim \left( \int_x^\infty w \right)^{\frac{1}{q}} \int_0^x k(x, z) f(z) dz, \end{aligned}$$

the inequality (1.1) implies (2.1) and  $C_T \gtrsim A_0$ . Moreover,

$$\begin{aligned} (Tf)(x) &\geq \left( \int_x^\infty w(y) \left( \int_0^x k(y, z) f(z) dz \right)^q dy \right)^{\frac{1}{q}} \\ &\gtrsim \left( \int_x^\infty [k(y, x)]^q w(y) dy \right)^{\frac{1}{q}} \int_0^x f. \end{aligned}$$

Then (1.1) implies (2.2) and  $C_T \gtrsim A_1$ . It follows from (1.1) that

$$C_T \|f\|_{L_v^p} \geq \left( \int_0^t u \right)^{\frac{1}{r}} \|H_t f\|_{L_w^q}, \quad f \in \mathfrak{M}^+$$

for any  $t \in (0, \infty)$ . Hence,

$$C_T \geq \sup_{t \in (0, \infty)} \left( \int_0^t u \right)^{\frac{1}{r}} \|H_t\|_{L_v^p \rightarrow L_w^q}$$

and the lower bound  $C_T \gtrsim A_2$  is proved for  $p \leq r$ . Now, let  $r < p$ . We have

$$\begin{aligned} A_2^s &= \int_0^\infty u(x) \left( \int_0^x u \right)^{\frac{s}{p}} \|H_{\sigma^{-1}(x), \sigma(x)}\|_{L_v^p \rightarrow L_w^q}^s dx \\ &= \sum_{n \leq N} \int_{a_n}^{a_{n+1}} u(x) \left( \int_0^x u \right)^{\frac{s}{p}} \|H_{\sigma^{-1}(x), \sigma(x)}\|_{L_v^p \rightarrow L_w^q}^s dx \\ &\leq \sum_{n \leq N} \left( \int_{a_n}^{a_{n+1}} u \right) \left( \int_0^{a_{n+1}} u \right)^{\frac{s}{p}} \|H_{\sigma^{-1}(a_n), \sigma(a_{n+1})}\|_{L_v^p \rightarrow L_w^q}^s \\ &\approx \sum_{n \leq N} (2^n)^{\frac{s}{r}} \|H_{a_{n-1}, a_{n+2}}\|_{L_v^p \rightarrow L_w^q}^s =: \bar{A}_2^s. \end{aligned}$$

Let  $\theta \in (0, 1)$  be arbitrary. For all  $n \leq N$  there is  $f_n \in \mathfrak{M}^+$  such that  $\text{supp} f_n \subset [a_{n-2}, a_{n+2}]$ ,  $\|f_n\|_{L_v^p} = 1$  and

$$\|H_{a_{n-1}, a_{n+2}} f_n\|_{L_w^q} \geq \theta \|H_{a_{n-1}, a_{n+2}}\|_{L_v^p \rightarrow L_w^q}.$$

Put

$$g_n := (2^n)^{\frac{s}{pr}} \|H_{a_{n-1}, a_{n+2}}\|_{L_v^p \rightarrow L_w^q}^{\frac{s}{p}} f_n, \quad g := \sum_{n \leq N} g_n.$$

We find

$$\begin{aligned}
\|g\|_{L_v^p}^p &= \sum_{j \leq N} \int_{a_j}^{a_{j+1}} \left( \sum_{n \leq N} g_n(x) \right)^p v(x) dx \\
&= \sum_{j \leq N} \int_{a_j}^{a_{j+1}} \left( \sum_{n=j-1}^{j+2} g_n(x) \right)^p v(x) dx \\
&\lesssim \sum_{j \leq N} \int_{a_{j-2}}^{a_{j+2}} g_j(x)^p v(x) dx \\
&= \sum_{j \leq N} (2^j)^{\frac{s}{r}} \|H_{a_{j-1}, a_{j+2}}\|_{L_v^p \rightarrow L_w^q}^s = \bar{A}_2^s.
\end{aligned}$$

Finally, applying (1.1)

$$\begin{aligned}
C_T^r \bar{A}_2^{\frac{sr}{p}} &\gtrsim C_T^r \|g\|_{L_v^p}^r \geq \int_0^\infty [Tg]^r u \geq \sum_{n \leq N} \int_{a_{n-2}}^{a_{n-1}} [Tg]^r u \\
&\geq \sum_{n \leq N} \left( \int_{a_{n-2}}^{a_{n-1}} u \right) \|H_{a_{n-1}, a_{n+2}} g\|_{L_w^q}^r \gtrsim \sum_{n \leq N} 2^n \|H_{a_{n-1}, a_{n+2}} g_n\|_{L_w^q}^r \\
&= \sum_{n \leq N} (2^n)^{\frac{s}{r}} \|H_{a_{n-1}, a_{n+2}}\|_{L_v^p \rightarrow L_w^q}^{\frac{sr}{p}} \|H_{a_{n-1}, a_{n+2}} f_n\|_{L_w^q}^r \\
&\geq \theta^r \sum_{n \leq N} (2^n)^{\frac{s}{r}} \|H_{a_{n-1}, a_{n+2}}\|_{L_v^p \rightarrow L_w^q}^s \gtrsim \theta^r \bar{A}_2^s.
\end{aligned}$$

Thus,  $C_T \gtrsim \theta \bar{A}_2$ . Hence,  $C_T \gtrsim \theta A_2$  and the required lower bound  $C_T \gtrsim A_0 + A_1 + A_2$  follows.

The case  $q = \infty$  is treated similarly with only replacement of (2.6) by a trivial modification

$$\sum_{n \in \mathbb{Z}} 2^n \left( \sup_{i \geq n} \lambda_i \right)^s \approx \sum_{n \in \mathbb{Z}} 2^n \lambda_n^s. \quad (2.7)$$

□

**Remark 2.2.** For  $p = \infty$  we have

$$C_T = \left\| T \left( \frac{1}{v} \right) \right\|_{L_u^r} \quad (2.8)$$

and for  $r = \infty$

$$C_T = \sup_{t \geq 0} U(t) \|H_t\|_{L_v^p \rightarrow L_w^q}, \quad (2.9)$$

where  $U(t) := \operatorname{ess\,sup}_{0 \leq x \leq t} u(x)$ .

Now, for  $0 < c < d \leq \infty$  and  $f \in \mathfrak{M}^+$  we put

$$\begin{aligned}
(H_{c,d}^* f)(x) &:= \chi_{[c,d)}(x) \int_x^{\sigma(d)} k(z, x) f(z) dz, \\
(H_c^* f)(x) &:= \chi_{[c,\infty)}(x) \int_x^\infty k(z, x) f(z) dz.
\end{aligned}$$

**Theorem 2.3.** Let  $1 \leq p < \infty$ ,  $0 < r < \infty$ ,  $0 < q \leq \infty$ ,  $\frac{1}{s} := \left(\frac{1}{r} - \frac{1}{p}\right)_+$ . Then the inequality (1.3) is fulfilled if and only if the inequalities

$$\left( \int_0^\infty u(x) \left( \int_x^{\sigma^2(x)} w \right)^{\frac{r}{q}} \left( \int_{\sigma^2(x)}^\infty k(z, \sigma^2(x)) f(z) dz \right)^r dx \right)^{\frac{1}{r}} \leq \mathbb{A}_0 \|f\|_{L_v^p}, \quad (2.10)$$

$$\left( \int_0^\infty u(x) \left( \int_x^{\sigma^2(x)} [k(\sigma^2(x), z)]^q w(z) dz \right)^{\frac{r}{q}} \left( \int_{\sigma^2(x)}^\infty f \right)^r dx \right)^{\frac{1}{r}} \leq \mathbb{A}_1 \|f\|_{L_v^p}, \quad (2.11)$$

if  $q < \infty$  or

$$\left( \int_0^\infty u(x) \left[ \operatorname{ess\,sup}_{y \in (x, \sigma^2(x))} w(y) \right]^r \left( \int_{\sigma^2(x)}^\infty k(z, \sigma^2(x)) f(z) dz \right)^r dx \right)^{\frac{1}{r}} \leq \mathbb{A}_0 \|f\|_{L_v^p}, \quad (2.12)$$

$$\left( \int_0^\infty u(x) \left[ \operatorname{ess\,sup}_{y \in (x, \sigma^2(x))} [w(y) k(\sigma^2(x), y)]^r \left( \int_{\sigma^2(x)}^\infty f \right)^r dx \right)^{\frac{1}{r}} \leq \mathbb{A}_1 \|f\|_{L_v^p} \quad (2.13)$$

for  $q = \infty$  hold for all  $f \in \mathfrak{M}^+$  and the constant

$$\mathbb{A}_2 := \begin{cases} \sup_{t \in (0, \infty)} \left( \int_0^t u \right)^{\frac{1}{r}} \|H_t^*\|_{L_v^p \rightarrow L_w^q}, & p \leq r, \\ \left( \int_0^\infty u(x) \left( \int_0^x u \right)^{\frac{s}{p}} \|H_{\sigma^{-1}(x), \sigma(x)}^*\|_{L_v^p \rightarrow L_w^q}^s dx \right)^{\frac{1}{s}}, & r < p \end{cases} \quad (2.14)$$

is finite. Moreover,  $C_S \approx \mathbb{A}_0 + \mathbb{A}_1 + \mathbb{A}_2$ .

*Proof.* Let the sequence  $\{a_n\}$  be the same as in the proof of Theorem 2.1 and  $q < \infty$ .

*Sufficiency.* We have

$$\begin{aligned} \int_0^\infty [Sf]^r u &= \sum_{n \leq N} \int_{\Delta_n} [Sf]^r u \approx \sum_{n \leq N} 2^n \left( \int_{a_n}^\infty w(y) \left( \int_y^\infty k(z, y) f(z) dz \right)^q dy \right)^{\frac{r}{q}} \\ &\approx \sum_{n \leq N} 2^n \left( \int_{\Delta_n} w(y) \left( \int_y^\infty k(z, y) f(z) dz \right)^q dy \right)^{\frac{r}{q}} \\ &\approx \sum_{n \leq N} 2^n \left( \int_{\Delta_n} w(y) \left( \int_y^{a_{n+2}} k(z, y) f(z) dz \right)^q dy \right)^{\frac{r}{q}} \\ &+ \sum_{n \leq N} 2^n \left( \int_{\Delta_n} w(y) \left( \int_{a_{n+2}}^\infty k(z, y) f(z) dz \right)^q dy \right)^{\frac{r}{q}} =: I_1^r + I_2^r. \end{aligned}$$

Since for  $y \in \Delta_n$ ,  $x \in \Delta_{n-1}$ ,  $z \in (a_{n+2}, \infty)$  it holds  $k(z, y) \approx k(z, \sigma^2(x)) + k(\sigma^2(x), y)$ , then the term  $I_2^r$  is estimated as follows



$$\begin{aligned}
I_2^r &\lesssim \sum_{n \leq N} \int_{a_{n-1}}^{a_n} u(x) \left( \int_{\Delta_n} w(y) \left( \int_{a_{n+2}}^{\infty} k(z, y) f(z) dz \right)^q dy \right)^{\frac{r}{q}} dx \\
&\lesssim \sum_{n \leq N} \int_{a_{n-1}}^{a_n} u(x) \left( \int_{\Delta_n} w \right)^{\frac{r}{q}} \left( \int_{\sigma^2(x)}^{\infty} k(z, \sigma^2(x)) f(z) dz \right)^r dx \\
&\quad + \sum_{n \leq N} \int_{a_{n-1}}^{a_n} u(x) \left( \int_{\Delta_n} [k(\sigma^2(x), y)]^q w(y) dy \right)^{\frac{r}{q}} \left( \int_{\sigma^2(x)}^{\infty} f \right)^r dx \\
&\lesssim \sum_{n \leq N} \int_{a_{n-1}}^{a_n} u(x) \left( \int_x^{\sigma^2(x)} w \right)^{\frac{r}{q}} \left( \int_{\sigma^2(x)}^{\infty} k(z, \sigma^2(x)) f(z) dz \right)^r dx \\
&\quad + \sum_{n \leq N} \int_{a_{n-1}}^{a_n} u(x) \left( \int_x^{\sigma^2(x)} [k(\sigma^2(x), y)]^q w(y) dy \right)^{\frac{r}{q}} \left( \int_{\sigma^2(x)}^{\infty} f \right)^r dx \\
&\leq (\mathbb{A}_0^r + \mathbb{A}_1^r) \left( \int_0^{\infty} f^p v \right)^{\frac{r}{p}}.
\end{aligned}$$

To estimate  $I_1^r$  we write

$$I_1^r \lesssim \sum_{n \leq N} \left( \int_{a_{n-1}}^{a_n} u \right) \|H_{a_n, a_{n+1}}^*\|_{L_v^p \rightarrow L_w^q}^r \left( \int_{a_n}^{a_{n+2}} f^p v \right)^{\frac{r}{p}}.$$

If  $p \leq r$ , by Jensen's inequality

$$I_1 \lesssim \sup_{n \leq N} \left( \int_{a_{n-1}}^{a_n} u \right)^{\frac{1}{r}} \|H_{a_n, a_{n+1}}^*\|_{L_v^p \rightarrow L_w^q} \|f\|_{L_v^p} \leq \mathbb{A}_2 \|f\|_{L_v^p}.$$

If  $r < p$ , applying Hölder's inequality with exponents  $\frac{s}{r}$  and  $\frac{p}{r}$  similar to the proof of Theorem 2.1, we find

$$\begin{aligned}
I_1^s &\lesssim \sum_{n \leq N} \left( \int_{a_n}^{a_{n+1}} u \right)^{\frac{s}{r}} \|H_{a_n, a_{n+1}}^*\|_{L_v^p \rightarrow L_w^q}^s \|f\|_{L_v^p}^s \\
&\lesssim \sum_{n \leq N} \left( \int_{a_n}^{a_{n+1}} u(x) \left( \int_0^x u \right)^{\frac{s}{p}} \|H_{\sigma^{-1}(x), \sigma(x)}^*\|_{L_v^p \rightarrow L_w^q}^s dx \right) \|f\|_{L_v^p}^s \\
&\leq \mathbb{A}_2^s \|f\|_{L_v^p}^s.
\end{aligned}$$

Thus,  $C_S \lesssim \mathbb{A}_0 + \mathbb{A}_1 + \mathbb{A}_2$ .

*Necessity.* Since

$$(Sf)(x) \gtrsim \|\chi_{[x, \sigma^2(x))}\|_{L_w^q} \int_{\sigma^2(x)}^{\infty} k(z, \sigma^2(x)) f(z) dz,$$

then (1.3) implies (2.10) and  $C_S \gtrsim \mathbb{A}_0$ . Also,

$$\begin{aligned} (Sf)(x) &\geq \left( \int_x^{\sigma^2(x)} w(y) \left( \int_{\sigma^2(x)}^\infty k(z, y) f(z) dz \right)^q \right)^{\frac{1}{q}} \\ &\gtrsim \left( \int_x^{\sigma^2(x)} w(y) k(\sigma^2(x), y) dy \right)^{\frac{1}{q}} \int_{\sigma^2(x)}^\infty f. \end{aligned}$$

Therefore, (1.3) implies (2.11) and  $C_S \gtrsim \mathbb{A}_1$ .

Now, let  $t \in (0, \infty)$  be fixed. It follows from (1.3)

$$C_S \|f\|_{L_v^p} \geq \left( \int_0^t u \right)^{\frac{1}{r}} \|H_t^* f\|_{L_w^q}, \quad f \in \mathfrak{M}^+.$$

Hence,

$$C_S \geq \sup_{t \in (0, \infty)} \left( \int_0^t u \right)^{\frac{1}{r}} \|H_t^*\|_{L_v^p \rightarrow L_w^q}$$

and  $C_S \gtrsim \mathbb{A}_2$  for  $p \leq r$  is shown.

Now, let  $r < p$ . As in the proof of Theorem 2.1 we find

$$\begin{aligned} \mathbb{A}_2^s &= \int_0^\infty u(x) \left( \int_0^x u \right)^{\frac{s}{p}} \|H_{\sigma^{-1}(x), \sigma(x)}^*\|_{L_v^p \rightarrow L_w^q}^s dx \\ &\lesssim \sum_{n \leq N} (2^n)^{\frac{s}{r}} \|H_{a_{n-1}, a_{n+2}}^*\|_{L_v^p \rightarrow L_w^q}^s. \end{aligned}$$

Let  $\theta \in (0, 1)$  be arbitrary. For all  $n \leq N$  there is  $f_n \in \mathfrak{M}^+$  such that  $\text{supp} f_n \subset [a_{n-1}, a_{n+3}]$ ,  $\|f_n\|_{L_v^p} = 1$  and

$$\|H_{a_{n-1}, a_{n+2}}^* f_n\|_{L_w^q} \geq \theta \|H_{a_{n-1}, a_{n+2}}^*\|_{L_v^p \rightarrow L_w^q}.$$

Put

$$g_n := (2^n)^{\frac{s}{pr}} \|H_{a_{n-1}, a_{n+2}}^*\|_{L_v^p \rightarrow L_w^q}^{\frac{s}{p}} f_n, \quad g := \sum_{n \leq N} g_n.$$

Then

$$\begin{aligned} \|g\|_{L_v^p}^p &= \sum_{j \leq N} \int_{a_j}^{a_{j+1}} \left( \sum_{n \leq N} g_n(x) \right)^p v(x) dx \\ &= \sum_{j \leq N} \int_{a_j}^{a_{j+1}} \left( \sum_{n=j-2}^{j+1} g_n(x) \right)^p v(x) dx \\ &\lesssim \sum_{j \leq N} \int_{a_{j-1}}^{a_{j+3}} g_j(x)^p v(x) dx = \sum_{j \leq N} (2^j)^{\frac{s}{r}} \|H_{a_{j-1}, a_{j+2}}^*\|_{L_v^p \rightarrow L_w^q}^s. \end{aligned}$$

Now,

$$\int_0^\infty [Sg]^r u \gtrsim \theta^r \sum_{n \leq N} (2^n)^{\frac{s}{r}} \|H_{a_{n-1}, a_{n+2}}^*\|_{L_v^p \rightarrow L_w^q}^s$$

and we obtain  $C_S \gtrsim \mathbb{A}_2$  for  $r < p$  and  $C_S \gtrsim \mathbb{A}_0 + \mathbb{A}_1 + \mathbb{A}_2$  similar to the proof of Theorem 2.1. The case  $q = \infty$  is proved analogously.  $\square$

**Remark 2.4.** Precise characterization of the inequalities (2.1)-(2.4), (2.10)-(2.13), sharp estimates of the norms  $\|H_t\|_{L_v^p \rightarrow L_w^q}$ ,  $\|H_{\sigma^{-1}(x), \sigma(x)}\|_{L_v^p \rightarrow L_w^q}$ ,  $\|H_t^*\|_{L_v^p \rightarrow L_w^q}$  and  $\|H_{\sigma^{-1}(x), \sigma(x)}^*\|_{L_v^p \rightarrow L_w^q}$  are known and can be found (in various, but equivalent forms) by using, for instance, the results of [40], [41] and [26], where an integral form of criterion for the case  $0 < q < 1$  was found.

**Remark 2.5.** For  $p = \infty$  we have

$$C_S = \left\| S \left( \frac{1}{v} \right) \right\|_{L_u^r} \quad (2.15)$$

and for  $r = \infty$

$$C_S = \sup_{t \geq 0} U(t) \|H_t^*\|_{L_v^p \rightarrow L_w^q}, \quad (2.16)$$

where  $U(t) := \operatorname{ess\,sup}_{0 \leq x \leq t} u(x)$ .

### 3 Operators $\mathcal{T}$ and $\mathcal{S}$

For finding criteria for (1.2) and (1.4) we suppose that  $0 < \int_t^\infty u < \infty$  for all  $t > 0$  and define the functions  $\zeta : [0, \infty) \rightarrow [0, \infty)$ ,  $\zeta^{-1} : [0, \infty) \rightarrow [0, \infty)$  by

$$\begin{aligned} \zeta(x) &:= \sup \left\{ y > 0 : \int_y^\infty u \geq \frac{1}{2} \int_x^\infty u \right\}, \\ \zeta^{-1}(x) &:= \sup \left\{ y > 0 : \int_y^\infty u \geq 2 \int_x^\infty u \right\}, \end{aligned}$$

where  $\sup \emptyset = 0$ . Let, also,  $\zeta^m$ ,  $m \in \mathbb{N}$  be a composition of  $m$  functions  $\zeta$  and similar for  $\zeta^{-m}$ . For  $0 \leq c < d < \infty$  and  $f \in \mathfrak{M}^+$  put

$$\begin{aligned} (\mathcal{H}_{c,d}f)(x) &:= \chi_{(c,d]}(x) \int_x^{\zeta(d)} k(z,x) f(z) dz, \\ (\mathcal{H}_d f)(x) &:= \chi_{(0,d]}(x) \int_x^\infty k(z,x) f(z) dz, \\ (\mathcal{H}_{c,d}^* f)(x) &:= \chi_{(c,d]}(x) \int_{\zeta^{-1}(c)}^x k(x,z) f(z) dz, \\ (\mathcal{H}_d^* f)(x) &:= \chi_{(0,d]}(x) \int_0^x k(x,z) f(z) dz. \end{aligned}$$

Similar to the previous section we prove the following theorems.

**Theorem 3.1.** *Let  $1 \leq p < \infty$ ,  $0 < r < \infty$ ,  $0 < q \leq \infty$ ,  $\frac{1}{s} := \left( \frac{1}{r} - \frac{1}{p} \right)_+$ . For validity of the inequality (1.2) it is necessary and sufficient that the inequalities*

$$\begin{aligned} \left( \int_0^\infty u(x) \left( \int_0^x w \right)^{\frac{r}{q}} \left( \int_x^\infty k(z,x) f(z) dz \right)^r dx \right)^{\frac{1}{r}} &\leq \mathcal{A}_0 \|f\|_{L_v^p}, \\ \left( \int_0^\infty u(x) \left( \int_0^x [k(x,y)]^q w(y) dy \right)^{\frac{r}{q}} \left( \int_x^\infty f \right)^r dx \right)^{\frac{1}{r}} &\leq \mathcal{A}_1 \|f\|_{L_v^p}, \end{aligned} \quad (3.1)$$

if  $q < \infty$  or

$$\begin{aligned} & \left( \int_0^\infty u(x) [\operatorname{ess\,sup}_{y \in (0,x)} w(y)]^r \left( \int_x^\infty k(z,x) f(z) dz \right)^r dx \right)^{\frac{1}{r}} \leq \mathcal{A}_0 \|f\|_{L_v^p}, \\ & \left( \int_0^\infty u(x) [\operatorname{ess\,sup}_{y \in (0,x)} [w(y)k(x,y)]]^r \left( \int_x^\infty f \right)^r dx \right)^{\frac{1}{r}} \leq \mathcal{A}_1 \|f\|_{L_v^p} \end{aligned}$$

for  $q = \infty$  hold for all  $f \in \mathfrak{M}^+$  and the constant

$$\mathcal{A}_2 := \begin{cases} \sup_{t \in (0,\infty)} \left( \int_t^\infty u \right)^{\frac{1}{r}} \|\mathcal{H}_t\|_{L_v^p \rightarrow L_w^q}, & p \leq r, \\ \left( \int_0^\infty u(x) \left( \int_x^\infty u \right)^{\frac{s}{p}} \|\mathcal{H}_{\zeta^{-1}(x), \zeta(x)}\|_{L_v^p \rightarrow L_w^q}^s dx \right)^{\frac{1}{s}}, & r < p, \end{cases} \quad (3.2)$$

is finite. Moreover,  $C_{\mathcal{T}} \approx \mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2$ .

**Theorem 3.2.** Let  $1 \leq p < \infty$ ,  $0 < r < \infty$ ,  $0 < q \leq \infty$ ,  $\frac{1}{s} := \left(\frac{1}{r} - \frac{1}{p}\right)_+$ . For validity of the inequality (1.4) it is necessary and sufficient that the inequalities

$$\begin{aligned} & \left( \int_0^\infty u(x) \left( \int_{\zeta^{-2}(x)}^x w \right)^{\frac{r}{q}} \left( \int_0^{\zeta^{-2}(x)} k(\zeta^{-2}(x), z) f(z) dz \right)^r dx \right)^{\frac{1}{r}} \leq \mathbf{A}_0 \|f\|_{L_v^p}, \\ & \left( \int_0^\infty u(x) \left( \int_{\zeta^{-2}(x)}^x w(y) [k(y, \zeta^{-2}(x))]^q dy \right)^{\frac{r}{q}} \left( \int_0^{\zeta^{-2}(x)} f \right)^r dx \right)^{\frac{1}{r}} \leq \mathbf{A}_1 \|f\|_{L_v^p}, \end{aligned} \quad (3.3)$$

if  $q < \infty$  or

$$\begin{aligned} & \left( \int_0^\infty u(x) [\operatorname{ess\,sup}_{y \in (\zeta^{-2}(x), x)} w(y)]^r \left( \int_0^{\zeta^{-2}(x)} k(\zeta^{-2}(x), z) f(z) dz \right)^r dx \right)^{\frac{1}{r}} \leq \mathbf{A}_0 \|f\|_{L_v^p}, \\ & \left( \int_0^\infty u(x) [\operatorname{ess\,sup}_{y \in (\zeta^{-2}(x), x)} [w(y)k(y, \zeta^{-2}(x))]]^r \left( \int_0^{\zeta^{-2}(x)} f \right)^r dx \right)^{\frac{1}{r}} \leq \mathbf{A}_1 \|f\|_{L_v^p} \end{aligned}$$

for  $q = \infty$  hold for all  $f \in \mathfrak{M}^+$  and the constant

$$\mathbf{A}_2 := \begin{cases} \sup_{t \in (0,\infty)} \left( \int_t^\infty u \right)^{\frac{1}{r}} \|\mathcal{H}_t^*\|_{L_v^p \rightarrow L_w^q}, & p \leq r, \\ \left( \int_0^\infty u(x) \left( \int_x^\infty u \right)^{\frac{s}{p}} \|\mathcal{H}_{\zeta^{-1}(x), \zeta(x)}^*\|_{L_v^p \rightarrow L_w^q}^s dx \right)^{\frac{1}{s}}, & r < p \end{cases} \quad (3.4)$$

is finite. Moreover,  $C_{\mathcal{S}} \approx \mathbf{A}_0 + \mathbf{A}_1 + \mathbf{A}_2$ .

## 4 Operators T and S

Let the functions  $\sigma$  and  $\sigma^{-1}$  be the same as in the Section 2. For  $0 < c < d \leq \infty$  and  $f \in \mathfrak{M}^+$  we put

$$\begin{aligned} (\mathbf{H}_{c,d}f)(x) &:= \chi_{[c,d)}(x) \int_{\sigma^{-1}(c)}^x f(z)dz, & (\mathbf{H}_cf)(x) &:= \chi_{[c,\infty)}(x) \int_0^x f(z)dz, \\ (\mathbf{H}_{c,d}^*f)(x) &:= \chi_{[c,d)}(x) \int_x^{\sigma(d)} f(z)dz, & (\mathbf{H}_c^*f)(x) &:= \chi_{[c,\infty)}(x) \int_x^\infty f(z)dz. \end{aligned}$$

**Theorem 4.1.** Let  $1 \leq p < \infty$ ,  $0 < r < \infty$ ,  $0 < q \leq \infty$ ,  $\frac{1}{s} := \left(\frac{1}{r} - \frac{1}{p}\right)_+$ . For validity of the inequality (1.5) it is necessary and sufficient that

$$B := B_0 + B_1 + B_2 < \infty, \quad (4.1)$$

where  $B_0$  and  $B_1$  are the least possible constants in the inequalities

$$\left( \int_0^\infty u(x) \left( \int_x^\infty k(y,x)w(y)dy \right)^{\frac{r}{q}} \left( \int_0^x f \right)^r dx \right)^{\frac{1}{r}} \leq B_0 \|f\|_{L_v^p}, \quad (4.2)$$

and

$$\left( \int_0^\infty u(x) [k(\sigma^2(x), x)]^{\frac{r}{q}} \left( \int_{\sigma^2(x)}^\infty w(y) \left( \int_0^y f \right)^q dy \right)^{\frac{r}{q}} dx \right)^{\frac{1}{r}} \leq B_1 \|f\|_{L_v^p}, \quad (4.3)$$

if  $q < \infty$  or

$$\left( \int_0^\infty u(x) [\operatorname{ess\,sup}_{y \geq x} k(y,x)w(y)]^r \left( \int_0^x f \right)^r dx \right)^{\frac{1}{r}} \leq B_0 \|f\|_{L_v^p}, \quad (4.4)$$

$$\left( \int_0^\infty u(x) [k(\sigma^2(x), x)]^r \left( \operatorname{ess\,sup}_{y \geq \sigma^2(x)} w(y) \int_0^y f \right)^r dx \right)^{\frac{1}{r}} \leq B_1 \|f\|_{L_v^p}, \quad (4.5)$$

for  $q = \infty$  and  $B_2$  is defined by

$$B_2 := \begin{cases} \sup_{t>0} \left( \int_0^t u \right)^{\frac{1}{r}} \|\mathbf{H}_t\|_{L_v^p \rightarrow L_{w(\cdot)k(\cdot,t)}^q}, & p \leq r, \\ \left( \int_0^\infty u(x) \left( \int_0^x u \right)^{\frac{s}{p}} \|\mathbf{H}_{\sigma^{-1}(x), \sigma^2(x)}\|_{L_v^p \rightarrow L_{w(\cdot)k(\cdot, \sigma^{-1}(x))}^q}^s dx \right)^{\frac{1}{s}}, & r < p. \end{cases} \quad (4.6)$$

Moreover,  $C_T \approx B$ .

*Proof.* Let the sequence  $\{a_n\}$  be the same as in the proof of Theorem 2.1 and  $q < \infty$ .

*Sufficiency.* We write

$$\begin{aligned} J &:= \int_0^\infty [Tf]^r u = \sum_{n \leq N} \int_{a_n}^{a_{n+1}} [Tf]^r u \\ &\approx \sum_{n \leq N} 2^n \left( \int_{a_n}^\infty k(y, a_n)w(y) \left( \int_0^y f \right)^q dy \right)^{\frac{r}{q}} \approx J_1 + J_2, \end{aligned}$$

where

$$J_1 := \sum_{n \leq N} 2^n \left( \int_{a_n}^{a_{n+2}} k(y, a_n) w(y) \left( \int_0^y f \right)^q dy \right)^{\frac{r}{q}},$$

$$J_2 := \sum_{n \leq N} 2^n \left( \int_{a_{n+2}}^{\infty} k(y, a_n) w(y) \left( \int_0^y f \right)^q dy \right)^{\frac{r}{q}}.$$

*Estimate of  $J_1$ .* We have

$$J_1 \approx \sum_{n \leq N} 2^n \left( \int_{a_n}^{a_{n+2}} k(y, a_n) w(y) \left( \int_{a_{n-1}}^y f \right)^q dy \right)^{\frac{r}{q}}$$

$$+ \sum_{n \leq N} 2^n \left( \int_{a_n}^{a_{n+2}} k(y, a_n) w(y) dy \right)^{\frac{r}{q}} \left( \int_0^{a_{n-1}} f \right)^r = J_{1,1} + J_{1,2}.$$

For  $J_{1,2}$  we write

$$J_{1,2} \approx \sum_{n \leq N} \int_{a_{n-1}}^{a_n} u(x) dx \left( \int_{a_n}^{a_{n+2}} k(y, a_n) w(y) dy \right)^{\frac{r}{q}} \left( \int_0^{a_{n-1}} f \right)^r$$

$$\lesssim \int_0^{\infty} u(x) \left( \int_x^{\infty} k(y, x) w(y) dy \right)^{\frac{r}{q}} \left( \int_0^x f \right)^r dx \leq B_0^r \left( \int_0^{\infty} f^p v \right)^{\frac{r}{p}}.$$

For  $J_{1,1}$  we write

$$J_{1,1} \approx \sum_{n \leq N} 2^n \left( \int_{a_n}^{a_{n+2}} k(y, a_n) w(y) (\mathbf{H}_{a_n, a_{n+2}} f(y))^q dy \right)^{\frac{r}{q}}$$

$$\lesssim \sum_{n \leq N} 2^n \|\mathbf{H}_{a_n, a_{n+2}}\|_{L_v^p \rightarrow L_{w(\cdot)k(\cdot, a_n)}^q}^r \left( \int_{a_{n-1}}^{a_{n+2}} f^p v \right)^{\frac{r}{p}}.$$

If  $p \leq r$ , by Jensen's inequality we get

$$J_{1,1} \lesssim B_2^r \|f\|_{L_v^p}^r.$$

If  $r < p$ , by Hölder's inequality,

$$J_{1,1} \lesssim \left( \sum_{n \leq N} (2^n)^{\frac{s}{r}} \|\mathbf{H}_{a_n, a_{n+2}}\|_{L_v^p \rightarrow L_{w(\cdot)k(\cdot, a_n)}^q}^{\frac{s}{p}} \right)^{\frac{r}{s}} \|f\|_{L_v^p}^r$$

$$\lesssim \left( \sum_{n \leq N} \int_{a_n}^{a_{n+1}} u \left( \int_0^{a_n} u \right)^{\frac{s}{p}} \|\mathbf{H}_{\sigma^{-1}(a_{n+1}), \sigma^2(a_n)}\|_{L_v^p \rightarrow L_{w(\cdot)k(\cdot, \sigma^{-1}(a_{n+1}))}^q}^{\frac{s}{p}} \right)^{\frac{r}{s}} \|f\|_{L_v^p}^r$$

$$\lesssim B_2^r \|f\|_{L_v^p}^r.$$

Thus,

$$J_1 \lesssim (B_0 + B_2)^r \|f\|_{L_v^p}^r. \quad (4.7)$$

Estimate of  $J_2$ . Denote  $h(y) := w(y) \left( \int_0^y f \right)^q$  and using (1.9) we obtain

$$\begin{aligned} \int_{a_{n+2}}^{\infty} k(y, a_n) h(y) dy &= \sum_{i \geq n} \int_{a_{i+2}}^{a_{i+3}} k(y, a_n) h(y) dy \\ &\approx \sum_{i \geq n} \int_{a_{i+2}}^{a_{i+3}} k(y, a_{i+1}) h(y) dy + \sum_{i \geq n} \int_{a_{i+2}}^{a_{i+3}} k(a_{i+1}, a_n) h(y) dy \\ &\lesssim \sum_{i \geq n} \int_{a_{i+1}}^{a_{i+3}} k(y, a_{i+1}) h(y) dy + \sum_{i \geq n} \int_{a_{i+2}}^{a_{i+3}} k(a_{i+1}, a_n) h(y) dy =: I_{1,n} + I_{2,n}. \end{aligned}$$

Similar to the proof of (4.7) we find

$$\sum_{n \leq N} 2^n I_{1,n}^{\frac{r}{q}} \approx J_1 \lesssim (B_0 + B_2)^r \|f\|_{L_v^p}^r. \quad (4.8)$$

By [15], Lemma 3.1 there is  $\alpha \in (0, 1)$  such that

$$k(a_{i+1}, a_n) \lesssim \left( \sum_{j=n}^i [k(a_{j+1}, a_j)]^\alpha \right)^{\frac{1}{\alpha}}, \quad i \geq n. \quad (4.9)$$

By Minkowskii's inequality

$$\begin{aligned} I_{2,n} &\lesssim \sum_{i \geq n} \left( \sum_{j=n}^i [k(a_{j+1}, a_j)]^\alpha \right)^{\frac{1}{\alpha}} \int_{a_{i+2}}^{a_{i+3}} h(y) dy \\ &\leq \left( \sum_{j \geq n} [k(a_{j+1}, a_j)]^\alpha \left( \int_{a_{j+2}}^{\infty} h \right)^\alpha \right)^{\frac{1}{\alpha}}. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{n \leq N} 2^n I_{2,n}^{\frac{r}{q}} &\leq \sum_{n \leq N} 2^n \left( \sum_{j \geq n} [k(a_{j+1}, a_j)]^\alpha \left( \int_{a_{j+2}}^{\infty} h \right)^\alpha \right)^{\frac{r}{q\alpha}} \\ &\approx \sum_{n \leq N} 2^n k(a_{n+1}, a_n)^{\frac{r}{q}} \left( \int_{a_{n+2}}^{\infty} h \right)^{\frac{r}{q}} \\ &\approx \sum_{n \leq N} \left[ \int_{a_{n-1}}^{a_n} u \right] k(\sigma^2(a_{n-1}), a_n)^{\frac{r}{q}} \left( \int_{\sigma^2(a_n)}^{\infty} h \right)^{\frac{r}{q}} \\ &\lesssim \int_0^{\infty} u(x) k(\sigma^2(x), x)^{\frac{r}{q}} \left( \int_{\sigma^2(x)}^{\infty} w(y) \left( \int_0^y f \right)^q dy \right)^{\frac{r}{q}} dx \leq B_1^r \|f\|_{L_v^p}^r. \end{aligned}$$

In case of  $q = \infty$  we write

$$\begin{aligned} \operatorname{ess\,sup}_{y \in [a_{n+2}, \infty)} k(y, a_n) h(y) &= \sup_{i \geq n} \operatorname{ess\,sup}_{y \in [a_{i+2}, a_{i+3})} k(y, a_n) h(y) \\ &\lesssim \sup_{i \geq n} \operatorname{ess\,sup}_{y \in [a_{i+1}, a_{i+3})} k(y, a_{i+1}) h(y) + \sup_{i \geq n} k(a_{i+1}, a_n) \operatorname{ess\,sup}_{y \in [a_{i+2}, a_{i+3})} h(y) =: I_{1,n} + I_{2,n}. \end{aligned}$$

The estimate (4.8) follows by the same way. Also we have

$$\begin{aligned}
I_{2,n} &\lesssim \left( \sup_{i \geq n} \sum_{j=n}^i [k(a_{j+1}, a_j)]^\alpha \left( \operatorname{ess\,sup}_{y \in [a_{i+2}, a_{i+3}]} h(y) \right)^\alpha \right)^{\frac{1}{\alpha}} \\
&\leq \left( \sum_{j \geq n} [k(a_{j+1}, a_j)]^\alpha \sup_{i \geq n} \left( \operatorname{ess\,sup}_{y \in [a_{i+2}, a_{i+3}]} h(y) \right)^\alpha \chi_{[n,i]}(j) \right)^{\frac{1}{\alpha}} \\
&= \left( \sum_{j \geq n} [k(a_{j+1}, a_j)]^\alpha \left( \operatorname{ess\,sup}_{y \in [a_{j+2}, \infty)} h(y) \right)^\alpha \right)^{\frac{1}{\alpha}}
\end{aligned}$$

and the inequality  $\sum_{n \leq N} 2^n I_{2,n}^{\frac{r}{q}} \lesssim B_1^r \|f\|_{L_v^p}^r$  follows for this case too. Thus,

$$J_2 \lesssim (B_0 + B_1 + B_2)^r \|f\|_{L_v^p}^r$$

and the upper bound  $C_{\mathbf{T}} \lesssim B_0 + B_1 + B_2$  is proved.

*Necessity.* Suppose the inequality (1.5) hold, that is

$$\left( \int_0^\infty \left( \int_x^\infty k(y, x) w(y) \left( \int_0^y f \right)^q dy \right)^{\frac{r}{q}} u \right)^{\frac{1}{r}} \leq C_{\mathbf{T}} \left( \int_0^\infty f^p v \right)^{\frac{1}{p}} \quad (4.10)$$

for all  $f \in \mathfrak{M}^+$ . Narrowing the integration  $(0, y) \rightarrow (0, x)$  on the left-hand side, we see, that  $C_{\mathbf{T}} \geq B_0$ . Analogously, if  $(x, \infty) \rightarrow (\sigma^2(x), \infty)$ ,  $k(y, x) \gtrsim k(\sigma^2(x), x)$ , then  $C_{\mathbf{T}} \geq B_1$ . If  $(0, \infty) \rightarrow (0, t)$ ,  $(x, \infty) \rightarrow (t, \infty)$ ,  $k(y, x) \gtrsim k(y, t)$ , then

$$C_{\mathbf{T}} \geq \left( \int_0^t u \right)^{\frac{1}{r}} \|\mathbf{H}_t\|_{L_v^p \rightarrow L_{w(\cdot)k(\cdot, t)}^q} \quad (4.11)$$

for all  $t > 0$ . Consequently,  $C_{\mathbf{T}} \gtrsim B_2$  in case of  $p \leq r$ .

In the case  $r < p$  we write

$$\begin{aligned}
B_2^s &= \int_0^\infty u(x) \left( \int_0^x u \right)^{\frac{s}{p}} \|\mathbf{H}_{\sigma^{-1}(x), \sigma^2(x)}\|_{L_v^p \rightarrow L_{w(\cdot)k(\cdot, \sigma^{-1}(x))}^q}^s dx \\
&= \sum_{n \leq N} \int_{a_n}^{a_{n+1}} u(x) \left( \int_0^x u \right)^{\frac{s}{p}} \|\mathbf{H}_{\sigma^{-1}(x), \sigma^2(x)}\|_{L_v^p \rightarrow L_{w(\cdot)k(\cdot, \sigma^{-1}(x))}^q}^s dx \\
&\leq \sum_{n \leq N} \left( \int_{a_n}^{a_{n+1}} u \right) \left( \int_0^{a_{n+1}} u \right)^{\frac{s}{p}} \|\mathbf{H}_{\sigma^{-1}(a_n), \sigma^2(a_{n+1})}\|_{L_v^p \rightarrow L_{w(\cdot)k(\cdot, \sigma^{-1}(a_n))}^q}^s \\
&\approx \sum_{n \leq N} (2^n)^{\frac{s}{r}} \|\mathbf{H}_{a_{n-1}, a_{n+3}}\|_{L_v^p \rightarrow L_{w(\cdot)k(\cdot, a_{n-1})}^q}^s =: \bar{B}_2^s.
\end{aligned}$$

Let  $\theta \in (0, 1)$  be arbitrary. Then for all  $n \leq N$  there is  $f_n \in \mathfrak{M}^+$  such that  $\operatorname{supp} f_n \subset [a_{n-2}, a_{n+3}]$ ,  $\|f_n\|_{L_v^p} = 1$  and

$$\|\mathbf{H}_{a_{n-1}, a_{n+3}} f_n\|_{L_{w(\cdot)k(\cdot, a_{n-1})}^q} \geq \theta \|\mathbf{H}_{a_{n-1}, a_{n+3}}\|_{L_v^p \rightarrow L_{w(\cdot)k(\cdot, a_{n-1})}^q}.$$



Put

$$g_n := (2^n)^{\frac{s}{pr}} \|\mathbf{H}_{a_{n-1}, a_{n+3}}\|_{L_v^p \rightarrow L_{w(\cdot)k(\cdot, a_{n-1})}^q}^{\frac{s}{p}} f_n, \quad g := \sum_{n \leq N} g_n.$$

We have

$$\begin{aligned} \|g\|_{L_v^p}^p &= \sum_{j \leq N} \int_{a_j}^{a_{j+1}} \left( \sum_{n \leq N} g_n(x) \right)^p v(x) dx \\ &= \sum_{j \leq N} \int_{a_j}^{a_{j+1}} \left( \sum_{n=j-2}^{j+2} g_n(x) \right)^p v(x) dx \\ &\lesssim \sum_{j \leq N} \int_{a_{j-2}}^{a_{j+3}} g_j(x)^p v(x) dx \\ &= \sum_{j \leq N} (2^j)^{\frac{s}{r}} \|\mathbf{H}_{a_{j-1}, a_{j+3}}\|_{L_v^p \rightarrow L_{w(\cdot)k(\cdot, a_{j-1})}^q}^s = \bar{B}_2^s. \end{aligned}$$

Finally, applying (1.1)

$$\begin{aligned} C_{\mathbf{T}}^r \mathcal{A}^{\frac{sr}{p}} &\gtrsim C_{\mathbf{T}}^r \|g\|_{L_v^p}^r \geq \int_0^\infty [Tg]^r u \geq \sum_{n \leq N} \int_{a_{n-2}}^{a_{n-1}} [\mathbf{T}g]^r u \\ &\geq \sum_{n \leq N} \left( \int_{a_{n-2}}^{a_{n-1}} u \right) \|\mathbf{H}_{a_{n-1}, a_{n+3}} g\|_{L_{w(\cdot)k(\cdot, a_{n-1})}^q}^r \gtrsim \sum_{n \leq N} 2^n \|\mathbf{H}_{a_{n-1}, a_{n+3}} g_n\|_{L_{w(\cdot)k(\cdot, a_{n-1})}^q}^r \\ &= \sum_{n \leq N} (2^n)^{\frac{s}{r}} \|\mathbf{H}_{a_{n-1}, a_{n+3}}\|_{L_v^p \rightarrow L_{w(\cdot)k(\cdot, a_{n-1})}^q}^{\frac{sr}{p}} \|\mathbf{H}_{a_{n-1}, a_{n+3}} f_n\|_{L_{w(\cdot)k(\cdot, a_{n-1})}^q}^r \geq \theta^r \bar{B}_2^s. \end{aligned}$$

Thus,  $C_{\mathbf{T}} \gtrsim \theta \bar{B}_2$ . Hence,  $C_{\mathbf{T}} \gtrsim \theta B_2$  and the required lower bound  $C_{\mathbf{T}} \gtrsim B_0 + B_1 + B_2$  follows.  $\square$

**Remark 4.1.** Similar to (2.8) and (2.9) we have

$$C_{\mathbf{T}} = \left\| \mathbf{T} \left( \frac{1}{v} \right) \right\|_{L_u^r}, \quad p = \infty, \quad (4.12)$$

$$C_{\mathbf{T}} \approx \sup_{t \geq 0} U(t) \|\mathbf{H}_t\|_{L_v^p \rightarrow L_{w(\cdot)k(\cdot, t)}^q}, \quad r = \infty. \quad (4.13)$$

**Theorem 4.2.** Let  $1 \leq p < \infty$ ,  $0 < r < \infty$ ,  $0 < q \leq \infty$ ,  $\frac{1}{s} := \left( \frac{1}{r} - \frac{1}{p} \right)_+$ . For validity of the inequality (1.7) it is necessary and sufficient that

$$\mathbb{B} := \mathbb{B}_0 + \mathbb{B}_1 + \mathbb{B}_2 < \infty, \quad (4.14)$$

where  $\mathbb{B}_0$  and  $\mathbb{B}_1$  are the least possible constants in the inequalities

$$\left( \int_0^\infty u(x) \left( \int_x^{\sigma^3(x)} k(y, x) w(y) dy \right)^{\frac{r}{q}} \left( \int_{\sigma^3(x)}^\infty f \right)^r dx \right)^{\frac{1}{r}} \leq \mathbb{B}_0 \|f\|_{L_v^p}, \quad (4.15)$$

and

$$\left( \int_0^\infty u(x) k(\sigma^2(x), x)^{\frac{r}{q}} \left( \int_{\sigma^2(x)}^\infty w(y) \left( \int_y^\infty f \right)^q dy \right)^{\frac{r}{q}} dx \right)^{\frac{1}{r}} \leq \mathbb{B}_1 \|f\|_{L_v^p}, \quad (4.16)$$

when  $q < \infty$  and

$$\left( \int_0^\infty u(x) \left[ \operatorname{ess\,sup}_{x \leq y \leq \sigma^3(x)} k(y, x) w(y) \right]^r \left( \int_{\sigma^3(x)}^\infty f \right)^r dx \right)^{\frac{1}{r}} \leq \mathbb{B}_0 \|f\|_{L_v^p}, \quad (4.17)$$

and

$$\left( \int_0^\infty u(x) [k(\sigma^2(x), x)]^r \left( \operatorname{ess\,sup}_{y \geq \sigma^2(x)} w(y) \int_y^\infty f \right)^r dx \right)^{\frac{1}{r}} \leq \mathbb{B}_1 \|f\|_{L_v^p}, \quad (4.18)$$

if  $q = \infty$ . The constant  $\mathbb{B}_2$  is given by

$$\mathbb{B}_2 := \begin{cases} \sup_{t>0} \left( \int_0^t u \right)^{\frac{1}{r}} \|\mathbf{H}_t^*\|_{L_v^p \rightarrow L_{w(\cdot)k(\cdot, t)}^q}, & p \leq r, \\ \left( \int_0^\infty u(x) \left( \int_0^x u \right)^{\frac{s}{p}} \|\mathbf{H}_{\sigma^{-1}(x), \sigma^2(x)}^*\|_{L_v^p \rightarrow L_{w(\cdot)k(\cdot, \sigma^{-1}(x))}^q}^s dx \right)^{\frac{1}{s}}, & r < p. \end{cases} \quad (4.19)$$

Moreover,  $C_S \approx \mathbb{B}$ .

*Proof.* Let the sequence  $\{a_n\}$  be the same as in the proof of Theorem 4.1 and  $q < \infty$ .

*Sufficiency.* We write

$$\begin{aligned} J &:= \int_0^\infty [\mathbf{S}f]^r u = \sum_{n \leq N} \int_{a_n}^{a_{n+1}} [\mathbf{S}f]^r u \\ &\approx \sum_{n \leq N} 2^n \left( \int_{a_n}^\infty k(y, a_n) w(y) \left( \int_y^\infty f \right)^q dy \right)^{\frac{r}{q}} \approx J_1 + J_2, \end{aligned}$$

where

$$\begin{aligned} J_1 &:= \sum_{n \leq N} 2^n \left( \int_{a_n}^{a_{n+2}} k(y, a_n) w(y) \left( \int_y^\infty f \right)^q dy \right)^{\frac{r}{q}}, \\ J_2 &:= \sum_{n \leq N} 2^n \left( \int_{a_{n+2}}^\infty k(y, a_n) w(y) \left( \int_y^\infty f \right)^q dy \right)^{\frac{r}{q}}. \end{aligned}$$

*Estimate of  $J_1$ .* We have

$$\begin{aligned} J_1 &\approx \sum_{n \leq N} 2^n \left( \int_{a_n}^{a_{n+2}} k(y, a_n) w(y) \left( \int_y^{a_{n+3}} f \right)^q dy \right)^{\frac{r}{q}} \\ &\quad + \sum_{n \leq N} 2^n \left( \int_{a_n}^{a_{n+2}} k(y, a_n) w(y) dy \right)^{\frac{r}{q}} \left( \int_{a_{n+3}}^\infty f \right)^r = J_{1,1} + J_{1,2}. \end{aligned}$$

For  $J_{1,2}$  we write

$$\begin{aligned} J_{1,2} &\approx \sum_{n \leq N} \int_{a_{n-1}}^{a_n} u(x) dx \left( \int_{a_n}^{a_{n+2}} k(y, a_n) w(y) dy \right)^{\frac{r}{q}} \left( \int_{a_{n+3}}^{\infty} f \right)^r \\ &\lesssim \int_0^{\infty} u(x) \left( \int_x^{\sigma^3(x)} k(y, x) w(y) dy \right)^{\frac{r}{q}} \left( \int_{\sigma^3(x)}^{\infty} f \right)^r dx \leq \mathbb{B}_0^r \left( \int_0^{\infty} f^p v \right)^{\frac{r}{p}}. \end{aligned}$$

For  $J_{1,1}$  we write

$$\begin{aligned} J_{1,1} &\approx \sum_{n \leq N} 2^n \left( \int_{a_n}^{a_{n+2}} k(y, a_n) w(y) \left( \mathbf{H}_{a_n, a_{n+2}}^* f(y) \right)^q dy \right)^{\frac{r}{q}} \\ &\lesssim \sum_{n \leq N} 2^n \|\mathbf{H}_{a_n, a_{n+2}}^*\|_{L_v^p \rightarrow L_{w(\cdot)k(\cdot, a_n)}^q}^r \left( \int_{a_n}^{a_{n+3}} f^p v \right)^{\frac{r}{p}}. \end{aligned}$$

If  $p \leq r$  then, by Jensen's inequality, we get

$$J_{1,1} \lesssim \mathbb{B}_2^r \|f\|_{L_v^p}^r.$$

If  $r < p$  then, by Hölder's inequality,

$$\begin{aligned} J_{1,1} &\lesssim \left( \sum_{n \leq N} (2^n)^{\frac{s}{r}} \|\mathbf{H}_{a_n, a_{n+2}}^*\|_{L_v^p \rightarrow L_{w(\cdot)k(\cdot, a_n)}^q}^s \right)^{\frac{r}{s}} \|f\|_{L_v^p}^r \\ &\lesssim \left( \sum_{n \leq N} \int_{a_n}^{a_{n+1}} u \left( \int_0^{a_n} u \right)^{\frac{s}{p}} \|\mathbf{H}_{\sigma^{-1}(a_{n+1}), \sigma^2(a_n)}^*\|_{L_v^p \rightarrow L_{w(\cdot)k(\cdot, \sigma^{-1}(a_{n+1}))}^q}^s \right)^{\frac{r}{s}} \|f\|_{L_v^p}^r \\ &\lesssim \mathbb{B}_2^r \|f\|_{L_v^p}^r. \end{aligned}$$

Thus

$$J_1 \lesssim (\mathbb{B}_0 + \mathbb{B}_2)^r \|f\|_{L_v^p}^r. \quad (4.20)$$

*Estimate of  $J_2$ .* Denote  $h(y) := w(y) \left( \int_y^{\infty} f \right)^q$  and arguing similar to the proof of Theorem 4.1 we obtain

$$J_2 \lesssim (\mathbb{B}_0 + \mathbb{B}_1 + \mathbb{B}_2)^r \|f\|_{L_v^p}^r.$$

*Necessity.* Suppose that the inequality (1.3) holds, that is

$$\left( \int_0^{\infty} \left( \int_x^{\infty} k(y, x) w(y) \left( \int_y^{\infty} f \right)^q dy \right)^{\frac{r}{q}} u \right)^{\frac{1}{r}} \leq C_S \left( \int_0^{\infty} f^p v \right)^{\frac{1}{p}} \quad (4.21)$$

for all  $f \in \mathfrak{M}^+$ . Narrowing the integration  $(x, \infty) \rightarrow (x, \sigma^3(x))$  and  $(y, \infty) \rightarrow (\sigma^3(x), \infty)$  on the left-hand side, we see, that  $C_S \geq \mathbb{B}_0$ . Analogously, if  $(x, \infty) \rightarrow (\sigma^2(x), \infty)$ ,  $k(y, x) \gtrsim k(\sigma^2(x), x)$ , then  $C_S \geq \mathbb{B}_1$ . The proof of  $C_S \gtrsim \mathbb{B}_2$  is similar to the proof of  $C_T \gtrsim B_2$ .  $\square$

**Remark 4.2.** Similar to (2.15) and (2.16) the equalities

$$C_S = \left\| \mathbf{S} \left( \frac{1}{v} \right) \right\|_{L_u^r}, \quad p = \infty \quad (4.22)$$

and

$$C_S \approx \sup_{t \geq 0} U(t) \|\mathbf{H}_t^*\|_{L_v^p \rightarrow L_{w(\cdot)k(\cdot, t)}^q}, \quad r = \infty \quad (4.23)$$

hold true.

## 5 Operators $\mathfrak{T}$ and $\mathfrak{S}$

Let the functions  $\zeta, \zeta^{-1} : [0, \infty) \rightarrow [0, \infty)$  be the same as in the Section 3. For  $0 \leq c < d < \infty$  and  $f \in \mathfrak{M}^+$  we define operators

$$\begin{aligned} (\mathfrak{H}_{c,d}f)(x) &:= \chi_{(c,d]}(x) \int_x^{\zeta(d)} f(z) dz, \\ (\mathfrak{H}_d f)(x) &:= \chi_{(0,d]}(x) \int_x^\infty f(z) dz, \\ (\mathfrak{H}_{c,d}^* f)(x) &:= \chi_{(c,d]}(x) \int_{\zeta^{-1}(c)}^x f(z) dz, \\ (\mathfrak{H}_d^* f)(x) &:= \chi_{(0,d]}(x) \int_0^x f(z) dz. \end{aligned}$$

The following theorems are true.

**Theorem 5.1.** *Let  $1 \leq p < \infty$ ,  $0 < r < \infty$ ,  $0 < q \leq \infty$ ,  $\frac{1}{s} := \left(\frac{1}{r} - \frac{1}{p}\right)_+$ . For validity of the inequality (1.6) it is necessary and sufficient that the inequalities*

$$\begin{aligned} \left( \int_0^\infty u(x) \left( \int_0^x k(x,y) w(y) dy \right)^{\frac{r}{q}} \left( \int_x^\infty f \right)^r dx \right)^{\frac{1}{r}} &\leq \mathcal{B}_0 \|f\|_{L_v^p}, \\ \left( \int_0^\infty u(x) [k(x, \zeta^{-2}(x))]^{\frac{r}{q}} \left( \int_0^{\zeta^{-2}(x)} w(y) \left( \int_y^\infty f \right)^q dy \right)^{\frac{r}{q}} dx \right)^{\frac{1}{r}} &\leq \mathcal{B}_1 \|f\|_{L_v^p}, \end{aligned}$$

if  $q < \infty$  or

$$\begin{aligned} \left( \int_0^\infty u(x) [\text{ess sup}_{y \in (0,x)} k(x,y) w(y)]^r \left( \int_x^\infty f \right)^r dx \right)^{\frac{1}{r}} &\leq \mathcal{B}_0 \|f\|_{L_v^p}, \\ \left( \int_0^\infty u(x) [k(x, \zeta^{-2}(x))]^r \left( \text{ess sup}_{y \in (0, \zeta^{-2}(x))} w(y) \int_y^\infty f \right)^r dx \right)^{\frac{1}{r}} &\leq \mathcal{B}_1 \|f\|_{L_v^p} \end{aligned}$$

for  $q = \infty$  hold for all  $f \in \mathfrak{M}^+$  and the constant

$$\mathcal{B}_2 := \begin{cases} \sup_{t \in (0, \infty)} \left( \int_t^\infty u \right)^{\frac{1}{r}} \|\mathfrak{H}_t\|_{L_v^p \rightarrow L_{w(\cdot)k(t, \cdot)}^q}, & p \leq r, \\ \left( \int_0^\infty u(x) \left( \int_x^\infty u \right)^{\frac{s}{p}} \|\mathfrak{H}_{\zeta^{-1}(x), \zeta^2(x)}\|_{L_v^p \rightarrow L_{w(\cdot)k(\zeta^2(x), \cdot)}^q}^s dx \right)^{\frac{1}{s}}, & r < p, \end{cases}$$

is finite. Moreover,  $C_{\mathfrak{T}} \approx \mathcal{B}_0 + \mathcal{B}_1 + \mathcal{B}_2$ .

**Theorem 5.2.** *Let  $1 \leq p < \infty$ ,  $0 < r < \infty$ ,  $0 < q \leq \infty$ ,  $\frac{1}{s} := \left(\frac{1}{r} - \frac{1}{p}\right)_+$ . For validity of the inequality (1.8) it is necessary and sufficient that the inequalities*

$$\begin{aligned} & \left( \int_0^\infty u(x) \left( \int_{\zeta^{-3}(x)}^x k(x, y) w(y) dy \right)^{\frac{r}{q}} \left( \int_0^{\zeta^{-3}(x)} f \right)^r dx \right)^{\frac{1}{r}} \leq \mathbf{B}_0 \|f\|_{L_v^p}, \\ & \left( \int_0^\infty u(x) [k(x, \zeta^{-2}(x))]^{\frac{r}{q}} \left( \int_0^{\zeta^{-2}(x)} w(y) \left( \int_0^y f \right)^q dy \right)^{\frac{r}{q}} dx \right)^{\frac{1}{r}} \leq \mathbf{B}_1 \|f\|_{L_v^p}, \end{aligned}$$

if  $q < \infty$  or

$$\begin{aligned} & \left( \int_0^\infty u(x) \left[ \operatorname{ess\,sup}_{y \in (\zeta^{-3}(x), x)} k(x, y) w(y) \right]^r \left( \int_0^{\zeta^{-3}(x)} f \right)^r dx \right)^{\frac{1}{r}} \leq \mathbf{B}_0 \|f\|_{L_v^p}, \\ & \left( \int_0^\infty u(x) [k(x, \zeta^{-2}(x))]^r \left( \operatorname{ess\,sup}_{y \in (0, \zeta^{-2}(x))} w(y) \int_0^y f \right)^r dx \right)^{\frac{1}{r}} \leq \mathbf{B}_1 \|f\|_{L_v^p} \end{aligned}$$

for  $q = \infty$  hold for all  $f \in \mathfrak{M}^+$  and the constant

$$\mathbf{B}_2 := \begin{cases} \sup_{t \in (0, \infty)} \left( \int_t^\infty u \right)^{\frac{1}{r}} \|\mathfrak{H}_t^*\|_{L_v^p \rightarrow L_{w(\cdot)k(t, \cdot)}^q}, & p \leq r, \\ \left( \int_0^\infty u(x) \left( \int_x^\infty u \right)^{\frac{s}{p}} \|\mathfrak{H}_{\zeta^{-1}(x), \zeta^2(x)}^*\|_{L_v^p \rightarrow L_{w(\cdot)k(\zeta^2(x), \cdot)}^q}^s dx \right)^{\frac{1}{s}}, & r < p \end{cases}$$

is finite. Moreover,  $C_{\mathfrak{S}} \approx \mathbf{B}_0 + \mathbf{B}_1 + \mathbf{B}_2$ .

## 6 $\Gamma^p(v) \rightarrow \Gamma^q(w)$ boundedness of the maximal operator

The maximal Hardy-Littlewood operator is defined by

$$Mf(x) := \sup_B \frac{1}{\operatorname{mes} B} \int_B |f(y)| dy$$

where the supremum is taken over all balls centered at  $x \in \mathbb{R}^n$ . The Lorentz  $\Gamma$ -spaces were introduced by E.T. Sawyer [32] while working on characterization of the boundedness of the maximal operator in the weighted Lorentz spaces (see also, for instance, related papers [8], [9], [14], [15], [34], [38]). More exactly, if  $v \in \mathfrak{M}^+$  and  $0 < p < \infty$ , then

$$\Gamma^p(v) = \left\{ f \text{ measurable on } \mathbb{R}^n : \left( \int_0^\infty [f^{**}(x)]^p v(x) dx \right)^{\frac{1}{p}} < \infty \right\},$$

where  $f^{**}(x) := \frac{1}{x} \int_0^x f^*(t) dt$  and

$$f^*(t) := \inf\{s > 0 : \operatorname{mes}\{x : |f(x)| > s\} \leq t\}.$$

It is known ([1], Theorem 3.8) that

$$[Mf]^*(x) \approx \frac{1}{x} \int_0^x f^*.$$

Therefore,  $M : \Gamma^p(v) \rightarrow \Gamma^q(u)$  boundedness is equivalent to the weighted inequality

$$\left( \int_0^\infty \left( \frac{1}{x} \int_0^x \left( \frac{1}{y} \int_0^y f \right) dy \right)^q u(x) dx \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty \left( \frac{1}{t} \int_0^t f \right)^p v(t) dt \right)^{\frac{1}{p}}, \quad f \in \mathfrak{M}^\downarrow \quad (6.1)$$

restricted on the cone  $\mathfrak{M}^\downarrow \subset \mathfrak{M}^+$  of all nonincreasing functions. Moreover, the least possible constant  $C$  is equivalent to the norm of  $M$

$$C \approx \|M\|_{\Gamma^p(v) \rightarrow \Gamma^q(u)} := \sup_{0 \neq f \in \Gamma^p(v)} \frac{\|Mf\|_{\Gamma^q(u)}}{\|f\|_{\Gamma^p(v)}}.$$

The inequality (6.1) was first characterized in the case  $1 < p = q < \infty, u = v$  ([39], Theorem 5.1) and for  $1 < p, q < \infty, u \neq v$  ([10], Theorem 3.3) and ([33], Theorem 5.1) (see, also [11]).

Applying Theorems 3.1 and 3.2 we solve the problem for all  $0 < p, q < \infty$  and similar to [33] our criteria have an explicit integral form.

Let  $\Omega_{1,0} := \{g \in \mathfrak{M}^\downarrow, tg(t) \in \mathfrak{M}^\uparrow\}$ . Then  $F(t) = \frac{1}{t} \int_0^t f \in \Omega_{1,0}$  for any  $f \in \mathfrak{M}^\downarrow$  and  $F^p \in \Omega_{p,0} := \{g \in \mathfrak{M}^\downarrow, t^p g(t) \in \mathfrak{M}^\uparrow\}$ . By the change  $G = F^p$  (6.1) becomes equivalent to

$$\left( \int_0^\infty \left( \frac{1}{x} \int_0^x G^{\frac{1}{p}} \right)^q u(x) dx \right)^{\frac{p}{q}} \leq C^p \int_0^\infty Gv, \quad G \in \Omega_{p,0} \quad (6.2)$$

and applying ([33], Lemma 2.3) we reduce (6.2) to the inequality

$$\left( \int_0^\infty \left( \frac{1}{x} \int_0^x \left( \int_0^\infty \frac{h(z) dz}{y^p + z^p} \right)^{\frac{1}{p}} dy \right)^q u(x) dx \right)^{\frac{p}{q}} \lesssim C^p \int_0^\infty hV, \quad h \in \mathfrak{M}^+, \quad (6.3)$$

where

$$V(z) = \int_0^\infty \frac{v(y) dy}{y^p + z^p}.$$

Since

$$\int_0^\infty \frac{h(z) dz}{y^p + z^p} \approx \int_y^\infty \frac{h(z) dz}{z^p} + \frac{1}{y^p} \int_0^y h(z) dz,$$

(6.3) is characterized by the following pair of inequalities:

$$\left( \int_0^\infty \left( \frac{1}{x} \int_0^x \left( \int_y^\infty h(z) dz \right)^{\frac{1}{p}} dy \right)^q u(x) dx \right)^{\frac{p}{q}} \leq C_1^p \int_0^\infty h(t) t^p V(t) dt, \quad h \in \mathfrak{M}^+$$

and

$$\left( \int_0^\infty \left( \frac{1}{x} \int_0^x \left( \int_0^y h(z) dz \right)^{\frac{1}{p}} \frac{dy}{y} \right)^q u(x) dx \right)^{\frac{p}{q}} \leq C_2^p \int_0^\infty hV, \quad h \in \mathfrak{M}^+,$$

which are of the form (1.2) and (1.4), respectively. Moreover,

$$C \approx C_1 + C_2.$$

Hence, applying Theorems 3.1 and 3.2, we see that

$$C_1 \approx \mathscr{A}_0 + \mathscr{A}_2 \quad (6.4)$$

and

$$C_2 \approx \mathbf{A}_0 + \mathbf{A}_2, \quad (6.5)$$

where the constants  $\mathbf{A}$ 's are defined by (3.1) and (3.2) for (6.4) and by (3.3) and (3.4) for (6.5) under related changes of weights, the function  $\zeta$  and auxiliary operators.

Suppose for simplicity that  $0 < \int_t^\infty s^{-q} u(s) ds < \infty$  for all  $t > 0$ . Now, the functions  $\zeta$  and  $\zeta^{-1}$  are defined by

$$\begin{aligned} \zeta(x) &:= \sup \left\{ y > 0 : \int_y^\infty s^{-q} u(s) ds \geq \frac{1}{2} \int_x^\infty s^{-q} u(s) ds \right\}, \\ \zeta^{-1}(x) &:= \sup \left\{ y > 0 : \int_y^\infty s^{-q} u(s) ds \geq 2 \int_x^\infty s^{-q} u(s) ds \right\}. \end{aligned}$$

For  $0 \leq c < d < \infty$  and  $h \in \mathfrak{M}^+$  we put

$$\begin{aligned} (\mathcal{H}_{c,d} h)(x) &:= \chi_{(c,d]}(x) \int_x^{\zeta(d)} h, \\ (\mathcal{H}_d h)(x) &:= \chi_{(0,d]}(x) \int_x^\infty h, \\ (\mathcal{H}_{c,d}^* h)(x) &:= \chi_{(c,d]}(x) \int_{\zeta^{-1}(c)}^x h, \\ (\mathcal{H}_d^* h)(x) &:= \chi_{(0,d]}(x) \int_0^x h. \end{aligned}$$

By Theorem 3.1  $\mathcal{A}_0$  is the least possible constant in the inequality

$$\left( \int_0^\infty u(x) \left( \int_x^\infty h \right)^{\frac{q}{p}} dx \right)^{\frac{p}{q}} \leq \mathcal{A}_0^p \int_0^\infty h(z) z^p V(z) dz, \quad h \in \mathfrak{M}^+$$

and  $\mathcal{A}_2$  is defined by

$$\mathcal{A}_2^p := \begin{cases} \sup_{t \in (0, \infty)} \left( \int_t^\infty s^{-q} u(s) ds \right)^{\frac{p}{q}} \|\mathcal{H}_t\|_{L_{z^p V(z)}^1 \rightarrow L^{\frac{1}{p}}}, & p \leq q, \\ \left( \int_0^\infty x^{-q} u(x) \left( \int_x^\infty s^{-q} u(s) ds \right)^{\frac{q}{p-q}} \|\mathcal{H}_{\zeta^{-1}(x), \zeta(x)}\|_{L_{z^p V(z)}^1 \rightarrow L^{\frac{1}{p}}}^{\frac{q}{p-q}} dx \right)^{\frac{p-q}{q}}, & q < p. \end{cases} \quad (6.6)$$

Also, by Theorem 3.2  $\mathbf{A}_0$  is the best possible constant in the inequality

$$\left( \int_0^\infty x^{-q} u(x) \left( \log \frac{x}{\zeta^{-2}(x)} \right)^q \left( \int_0^{\zeta^{-2}(x)} h \right)^{\frac{q}{p}} dx \right)^{\frac{p}{q}} \leq \mathbf{A}_0^p \int_0^\infty h V, \quad h \in \mathfrak{M}^+$$

and  $\mathbf{A}_2$  is determined from

$$\mathbf{A}_2^p := \begin{cases} \sup_{t \in (0, \infty)} \left( \int_t^\infty s^{-q} u(s) ds \right)^{\frac{p}{q}} \|\mathcal{H}_t^*\|_{L_V^1 \rightarrow L^{\frac{1}{p}}}, & p \leq q, \\ \left( \int_0^\infty x^{-q} u(x) \left( \int_x^\infty s^{-q} u(s) ds \right)^{\frac{q}{p-q}} \|\mathcal{H}_{\zeta^{-1}(x), \zeta(x)}^*\|_{L_V^1 \rightarrow L^{\frac{1}{p}}}^{\frac{q}{p-q}} dx \right)^{\frac{p-q}{q}}, & q < p. \end{cases} \quad (6.7)$$

By well known results ( [18], Chapter XI, § 1.5, Theorem 4, see also [15], Theorem 1.1) and ( [35], Theorem 3.3) we have

$$\mathcal{A}_0^p = \sup_{t>0} \frac{\left(\int_0^t u\right)^{\frac{p}{q}}}{t^p V(t)}, \quad p \leq q \quad (6.8)$$

and

$$\mathcal{A}_0^p \approx \left( \int_0^\infty [t^p V(t)]^{\frac{q}{q-p}} \left( \int_0^t u \right)^{\frac{q}{p-q}} u(t) dt \right)^{\frac{p-q}{q}}, \quad q < p. \quad (6.9)$$

Analogously, we find

$$\mathbf{A}_0^p = \sup_{t>0} \frac{\left( \int_{\zeta^2(t)}^\infty x^{-q} u(x) \left( \log \frac{x}{\zeta^{-2}(x)} \right)^q dx \right)^{\frac{p}{q}}}{V(t)}, \quad p \leq q \quad (6.10)$$

and for  $q < p$

$$\mathbf{A}_0^p \approx \left( \int_0^\infty \left( \frac{\int_x^\infty s^{-q} u(s) \left( \log \frac{s}{\zeta^{-2}(s)} \right)^q ds}{V(\zeta^{-2}(x))} \right)^{\frac{q}{p-q}} x^{-q} u(x) \left( \log \frac{x}{\zeta^{-2}(x)} \right)^q dx \right)^{\frac{p-q}{q}}. \quad (6.11)$$

Again, applying ( [18], Chapter XI, § 1.5, Theorem 4) and ( [35], Theorem 3.3) we obtain

$$\|\mathcal{H}_t\|_{L_{z^p V(z)}^1 \rightarrow L^{\frac{1}{p}}} = [V(t)]^{-1}, \quad 0 < p \leq 1$$

and

$$\|\mathcal{H}_t\|_{L_{z^p V(z)}^1 \rightarrow L^{\frac{1}{p}}} \approx \left( \int_0^t [V(x)]^{\frac{1}{1-p}} \frac{dx}{x} \right)^{p-1}, \quad p > 1,$$

so that it follows from (6.6) for  $p \leq q$

$$\mathcal{A}_2 = \sup_{t \in (0, \infty)} \left( \int_t^\infty s^{-q} u(s) ds \right)^{\frac{1}{q}} [V(t)]^{-\frac{1}{p}}, \quad 0 < p \leq 1, \quad (6.12)$$

and

$$\mathcal{A}_2 \approx \sup_{t \in (0, \infty)} \left( \int_t^\infty s^{-q} u(s) ds \right)^{\frac{1}{q}} \left( \int_0^t [V(x)]^{\frac{1}{1-p}} \frac{dx}{x} \right)^{\frac{1}{p'}}, \quad p > 1, \quad (6.13)$$

where  $p' := \frac{p}{p-1}$ . By the same way,

$$\|\mathcal{H}_{\zeta^{-1}(x), \zeta(x)}\|_{L_{z^p V(z)}^1 \rightarrow L^{\frac{1}{p}}} = \left[ \frac{\zeta(x) - \zeta^{-1}(x)}{\zeta(x)} \right]^p \frac{1}{V(\zeta(x))}, \quad 0 < p \leq 1$$

and

$$\|\mathcal{H}_{\zeta^{-1}(x), \zeta(x)}\|_{L_{z^p V(z)}^1 \rightarrow L^{\frac{1}{p}}} \approx \left( \int_{\zeta^{-1}(x)}^{\zeta(x)} [t^p V(t)]^{\frac{1}{1-p}} (t - \zeta^{-1}(x))^{\frac{1}{p-1}} dt \right)^{p-1}, \quad p > 1.$$



Hence, from (6.6) we see that for  $q < p$

$$\mathcal{A}_2 \approx \left( \int_0^\infty x^{-q} u(x) \left( \int_x^\infty s^{-q} u(s) ds \right)^{\frac{q}{p-q}} \left[ \frac{(\zeta(x) - \zeta^{-1}(x))}{\zeta(x)[V(\zeta(x))]^{\frac{1}{p}}} \right]^{\frac{pq}{p-q}} dx \right)^{\frac{p-q}{pq}}, \quad (6.14)$$

if  $0 < p \leq 1$  and

$$\mathcal{A}_2 \approx \left( \int_0^\infty x^{-q} u(x) \left( \int_x^\infty s^{-q} u(s) ds \right)^{\frac{q}{p-q}} \left( \int_{\zeta^{-1}(x)}^{\zeta(x)} \left( \frac{t - \zeta^{-1}(x)}{t^p V(t)} \right)^{\frac{1}{p-1}} dt \right)^{\frac{q(p-1)}{p-q}} dx \right)^{\frac{p-q}{pq}}, \quad (6.15)$$

when  $p > 1$ .

Similarly,

$$\|\mathcal{H}_t^*\|_{L_V^1 \rightarrow L^{\frac{1}{\frac{1}{p}}}} = \sup_{s \in (0, t)} [V(s)]^{-1} \left( \log \frac{t}{s} \right)^p, \quad 0 < p \leq 1$$

and

$$\|\mathcal{H}_t^*\|_{L_V^1 \rightarrow L^{\frac{1}{\frac{1}{p}}}} \approx \left( \int_0^t [V(x)]^{\frac{1}{1-p}} \left( \log \frac{t}{x} \right)^{\frac{1}{p-1}} \frac{dx}{x} \right)^{p-1}, \quad p > 1.$$

Now, it follows from (6.7) for  $p \leq q$

$$\mathbf{A}_2 = \sup_{t \in (0, \infty)} \left( \int_t^\infty s^{-q} u(s) ds \right)^{\frac{1}{q}} \sup_{s \in (0, t)} [V(s)]^{-\frac{1}{p}} \log \frac{t}{s}, \quad 0 < p \leq 1 \quad (6.16)$$

and

$$\mathbf{A}_2 \approx \sup_{t \in (0, \infty)} \left( \int_t^\infty s^{-q} u(s) ds \right)^{\frac{1}{q}} \left( \int_0^t [V(x)]^{\frac{1}{1-p}} \left( \log \frac{t}{x} \right)^{\frac{1}{p-1}} \frac{dx}{x} \right)^{\frac{1}{p}}, \quad p > 1, \quad (6.17)$$

We have

$$\|\mathcal{H}_{\zeta^{-1}(x), \zeta(x)}^*\|_{L_V^1 \rightarrow L^{\frac{1}{\frac{1}{p}}}} = \sup_{s \in (\zeta^{-1}(x), \zeta(x))} \frac{\left( \log \frac{\zeta(x)}{s} \right)^p}{V(s)}, \quad 0 < p \leq 1$$

and

$$\|\mathcal{H}_{\zeta^{-1}(x), \zeta(x)}^*\|_{L_V^1 \rightarrow L^{\frac{1}{\frac{1}{p}}}} \approx \left( \int_{\zeta^{-1}(x)}^{\zeta(x)} [V(t)]^{\frac{1}{1-p}} \left( \log \frac{\zeta(x)}{t} \right)^{\frac{1}{p-1}} \frac{dt}{t} \right)^{p-1}, \quad p > 1.$$

Thus, from (6.7) we find for  $q < p$

$$\mathbf{A}_2 \approx \left( \int_0^\infty x^{-q} u(x) \left( \int_x^\infty s^{-q} u(s) ds \right)^{\frac{q}{p-q}} \left[ \sup_{s \in (\zeta^{-1}(x), \zeta(x))} \frac{\left( \log \frac{\zeta(x)}{s} \right)^p}{V(s)} \right]^{\frac{q}{p-q}} dx \right)^{\frac{p-q}{pq}}, \quad (6.18)$$

if  $0 < p \leq 1$  and

$$\mathbf{A}_2 \approx \left( \int_0^\infty x^{-q} u(x) \left( \int_x^\infty s^{-q} u(s) ds \right)^{\frac{q}{p-q}} \left( \int_{\zeta^{-1}(x)}^{\zeta(x)} \left( \frac{\log \frac{\zeta(x)}{t}}{V(t)} \right)^{\frac{1}{p-1}} \frac{dt}{t} \right)^{\frac{q(p-1)}{p-q}} dx \right)^{\frac{p-q}{pq}}, \quad (6.19)$$

when  $p > 1$ .

Finally, we obtain the following.

**Theorem 6.1.** *Let  $0 < p, q < \infty$ . Then for the maximal Hardy-Littlewood operator*

$$\|M\|_{\Gamma^p(v) \rightarrow \Gamma^q(u)} \approx \mathcal{A}_0 + \mathcal{A}_2 + \mathbf{A}_0 + \mathbf{A}_2, \quad (6.20)$$

where the constants on the right-hand side are determined by (6.8)-(6.11) for  $\mathcal{A}_0$  and  $\mathbf{A}_0$  and by (6.12)-(6.19) for  $\mathcal{A}_2$  and  $\mathbf{A}_2$ .

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